PHYSICAL REVIEW E **81**, 020102(R) (2010)

Some geometric critical exponents for percolation and the random-cluster model

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(Received 22 April 2009; revised manuscript received 13 November 2009; published 10 February 2010)

We introduce several infinite families of critical exponents for the random-cluster model and present scaling arguments relating them to the k-arm exponents. We then present Monte Carlo simulations confirming these predictions. These exponents provide a convenient way to determine k-arm exponents from Monte Carlo simulations. An understanding of these exponents also leads to a radically improved implementation of the Sweeny Monte Carlo algorithm. In addition, our Monte Carlo data allow us to conjecture an exact expression for the shortest-path fractal dimension $d_{\min}$ in two dimensions: $d_{\min} \stackrel{?}{=} (g+2)(g+18)/(32g)$, where g is the Coulomb-gas coupling, related to the cluster fugacity q via $q=2+2\cos(g\pi/2)$ with $2 \le g \le 4$.

DOI: 10.1103/PhysRevE.81.020102 PACS number(s): 64.60.De, 05.10.Ln, 05.50.+q, 64.60.ah

I. INTRODUCTION

The random-cluster model [1] is a correlated bond-percolation model that plays a central role in the theory of critical phenomena, especially in two dimensions where it arises in recent developments of conformal field theory [2] via its connection with Schramm-Loewner evolution (SLE) [3,4]. To each bond configuration $A \subseteq E$ of a given graph $G=(V,E)$, the random-cluster model assigns a weight proportional to $q^{k(A)}v^{|A|}$, where $k(A)$ is the number of clusters (connected components), $|A|$ is the number of bonds, and $v,q \ge 0$ are parameters (see Fig. 1). For $q=1$ the random-cluster model reduces to independent bond percolation [5], while for integer $q \ge 2$ it provides a graphical representation of the q-state ferromagnetic Potts model [6]. Furthermore, the $q \to 0$ limit corresponds to uniform spanning trees provided that $v \propto q^{\alpha}$ with $0<\alpha<1$; this applies in particular to the critical square-lattice random-cluster model, for which $v=\sqrt{q}$ [7]. The random-cluster model thus provides an extension of both percolation and the Potts model that allows all positive values of q, integer or noninteger, to be studied within a unified framework.

When the graph G is planar, it is useful to map the bond configurations to loop configurations [8], as illustrated in Fig. 1. The loop configurations are drawn on the medial graph, the vertices of which correspond to the edges of the original graph. The medial graph of the square lattice is again a square lattice, rotated 45°. Each unoccupied edge of the original lattice is crossed by precisely two loop arcs, while occupied edges are crossed by none. The continuum limits of such loops are of central interest in studies of SLE [4]. The outermost loop bounding a cluster defines its *hull* [9].

In this Rapid Communication we shall define and study several infinite families of critical exponents for the random-cluster model, related to cluster size, hull length, and shortest path. Some of these exponents have been studied previously for spanning trees or percolation only, while others appear to be entirely new. We shall give scaling arguments determining all but one of these exponents as a function of q in the two-dimensional (2D) case and Monte Carlo simulations confirming these predictions. The remaining undetermined exponent is the shortest-path fractal dimension $d_{\min}$ [10], which relates the shortest-path length and Euclidean distance between two vertices on a cluster. However, our Monte Carlo data lead us to conjecture the exact formula

$$d_{\min} \stackrel{?}{=} (g+2)(g+18)/(32g), \tag{1}$$

where $q=2+2\cos(g\pi/2)$ and $g \in [2,4]$ is the Coulomb-gas coupling [11]. To our knowledge, $d_{\min}$ has not previously been studied for $q \neq 0,1$.

Our original motivation for studying these exponents arose out of practical concerns related to Monte Carlo algorithms, as we discuss below. However, we subsequently discovered a relationship between them and the k-arm expo-

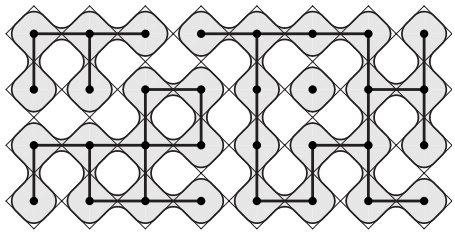

FIG. 1. Typical bond configuration $A \subseteq E$ on a finite subgraph of the square lattice: here $|A|=31$ and $k(A)=4$. Also shown are the corresponding medial-lattice loops (interiors shaded for clarity). Here $C_{\min,2}=|\mathcal{C}_{\mathbf{x}_1}|=11$; choosing $e_1=\mathbf{x}_1\mathbf{x}_2$ and $e_2=\mathbf{x}_2\mathbf{x}_3$ yields $M_{\min,2}=|\mathcal{M}_{\mathbf{x}_2,e_2}|=16$; and $T_{\min,2}=T_{\mathbf{x}_1}=4$. The graph-theoretic distance (shortest-path length) between $\mathbf{x}_2$ and $\mathbf{x}_3$ is 3.

nents x_k [9], which have proved to be of fundamental importance in critical phenomena, not least within the context of rigorous studies of percolation [12]. The exponents we study here are defined via the scaling of very natural graphical observables, which are easily measured in Monte Carlo simulations. Therefore, the relationship between these exponents and the k-arm exponents provides a new and convenient way to numerically estimate x_k, which can be used also in three dimensions where no exact expressions for x_k are known. This has allowed us, for example, to numerically verify in three-dimensional (3D) that x_2 equals the thermal dimension x_t when $q=1$, a result previously tested only in 2D.

II. MOTIVATION

Monte Carlo simulations are an essential tool in statistical mechanics, but they typically suffer from *critical slowing-down* [13]: the autocorrelation (relaxation) time τ diverges as a critical point is approached, most often as a power law $\tau \sim \xi^z$, where ξ is the spatial correlation length and z is a dynamic critical exponent.

The Sweeny algorithm [14] is a local single-bond update dynamics for the random-cluster model; for $0<q<1$ it is the *only* known general algorithm for this model. We have recently shown [15] that the Sweeny algorithm has unusually weak critical slowing-down, fairly close to the theoretical lower bound $z \geq \alpha/\nu$ [16]. Furthermore, it exhibits (especially for small q) the surprising phenomenon of "critical speeding-up" [15], in which suitable global observables exhibit significant decorrelation on time scales much *less* than one sweep (namely, L^w hits for some $w<d$); this makes the algorithm potentially very efficient.

The main obstacle to the use of the Sweeny algorithm is the need (when $q \neq 1$) to determine how updating a bond $\mathbf{xy}$ affects $k(A)$: this potentially requires traversing an entire cluster to determine whether $\mathbf{x}$ and $\mathbf{y}$ are connected. Given a lattice site $\mathbf{x}$, we write $\mathcal{C}_\mathbf{x}$ for the cluster containing $\mathbf{x}$ and $|\mathcal{C}_\mathbf{x}|$ for the number of sites in it. Since $|\mathcal{C}_\mathbf{x}|$ has mean $\sim L^{\gamma/\nu}$ near the critical point, the connectivity check threatens to impose a "computational critical slowing-down" that would more than outweigh the good "physical" behavior of the Sweeny dynamics. There do exist sophisticated algorithms in the computer-science literature [17] for performing such connectivity checking dynamically, which have been proven to be (asymptotically) very efficient, but their complexity appears prohibitive for use in practical simulations.

Our interest in the present project began with a simple idea for reducing this computational slowing-down without the need for complex data structures or algorithms: namely, perform simultaneous breadth-first searches starting at both end points $\mathbf{x}$ and $\mathbf{y}$, and stop when one of the clusters has been fully visited or the clusters merge. In the first case this takes a time $\min(|\mathcal{C}_\mathbf{x}|, |\mathcal{C}_\mathbf{y}|)$, and in the second case a time B_s (the number of sites visited in breadth-first search until merger). A natural question is therefore to determine the critical behavior of $C_{\min,2}$ and B_s, where $C_{\min,2} = \min(|\mathcal{C}_\mathbf{x}|, |\mathcal{C}_\mathbf{y}|)$ if $\mathcal{C}_\mathbf{x}$ and $\mathcal{C}_\mathbf{y}$ are distinct and 0 otherwise. We will provide here a scaling argument suggesting that both scale (in mean) as $L^{d_F-x_2} \ll L^{\gamma/\nu}$, where $d_F=d-\beta/\nu$ is the cluster fractal dimension and x_2 is the 2-arm exponent. We will then verify this prediction numerically.

In two dimensions an even more efficient procedure is to simultaneously follow the medial-lattice loops surrounding $\mathbf{x}$ and $\mathbf{y}$. If the two loops are distinct, this takes a time $\min(|\mathcal{M}_{\mathbf{x},\mathbf{xy}}|, |\mathcal{M}_{\mathbf{y},\mathbf{xy}}|)$, where $\mathcal{M}_{\mathbf{x},e}$ is the loop on the medial lattice that winds around $\mathbf{x}$ through the edge e if e is unoccupied, and $\mathcal{M}_{\mathbf{x},e}=\emptyset$ otherwise (see Fig. 1). This naturally leads to the question of the scaling of $M_{\min,2}$, defined analogously to $C_{\min,2}$ as $M_{\min,2}=\min(|\mathcal{M}_{\mathbf{x},\mathbf{xy}}|, |\mathcal{M}_{\mathbf{y},\mathbf{xy}}|)$ if $\mathcal{M}_{\mathbf{x},\mathbf{xy}}$ and $\mathcal{M}_{\mathbf{y},\mathbf{xy}}$ are distinct and 0 otherwise. We will provide a scaling argument, and confirm numerically, that $M_{\min,2}$ scales (in mean) as $\sim L^{d_H-x_2}$, where $d_H=1+2/g$ is the hull fractal dimension [9,18]. It follows that computational critical slowing down is completely absent for $q>4\cos^2(\pi\sqrt{2/3}) \approx 2.811\,520$.

III. DEFINITION OF EXPONENTS

These scaling results for $C_{\min,2}$ and $M_{\min,2}$ can be generalized in a very natural way. We shall consider a variety of positive-integer-valued observables $\mathcal{O}$; for each one we expect that its probability distribution obeys a scaling law $\mathbb{P}(\mathcal{O}=s) \sim s^{-\psi_\mathcal{O}}$ (with $\psi_\mathcal{O}>1$) for large s at criticality in infinite volume, or more generally $\sim s^{-\psi_\mathcal{O}} F_\mathcal{O}(s/\xi^{d_\mathcal{O}}, s/L^{d_\mathcal{O}})$ near criticality in large finite volume, where $F_\mathcal{O}$ is a scaling function. Our goal is to determine, for each $\mathcal{O}$, the decay exponent $\psi_\mathcal{O}$ and the fractal dimension $d_\mathcal{O}$. Note that at criticality in finite volume, $\langle \mathcal{O}^n \rangle \sim L^{(n+1-\psi_\mathcal{O})d_\mathcal{O}} + \text{const} + \text{corrections to scaling}$ as $L \to \infty$ (or $\sim \log L$ if $n+1-\psi_\mathcal{O}=0$); we shall use this fact in our Monte Carlo determinations of $\psi_\mathcal{O}$ and $d_\mathcal{O}$.

Fix nearby sites $\mathbf{x}_1, \ldots, \mathbf{x}_k$, and let $C_{\min,k} = \min(|\mathcal{C}_{\mathbf{x}_1}|, \ldots, |\mathcal{C}_{\mathbf{x}_k}|)$ if these clusters are all distinct and 0 otherwise. We expect that all these observables have fractal dimension $d_{C_{\min,k}}$ equal to the cluster fractal dimension $d_F = d-\beta/\nu$. Moreover, standard hyperscaling arguments [5] give $\psi_{C_{\min,1}} = d/d_F$ (the usual notation is $\tau = \psi_{C_{\min,1}} + 1$). Our goal is to determine $\psi_{C_{\min,k}}$ for $k \geq 2$. To our knowledge these exponents are new, except $\psi_{C_{\min,2}} = \frac{11}{8}$ for 2D spanning trees [19].

For two-dimensional lattices, choose for each site $\mathbf{x}_i$ a bond e_i incident on it, and let $M_{\min,k} = \min(|\mathcal{M}_{\mathbf{x}_1,e_1}|, \ldots, |\mathcal{M}_{\mathbf{x}_k,e_k}|)$ if these loops are all distinct and 0 otherwise. We expect that all these observables have fractal dimension $d_{M_{\min,k}}$ equal to the hull fractal dimension d_H; and standard hyperscaling arguments give $\psi_{M_{\min,1}} = d/d_H$. We aim to determine $\psi_{M_{\min,k}}$ for $k \geq 2$.

Now let $T_\mathbf{x}$ denote the maximum graph-theoretic distance from $\mathbf{x}$ to any site in $\mathcal{C}_\mathbf{x}$, and define $T_{\min,k} = \min(T_{\mathbf{x}_1}, \ldots, T_{\mathbf{x}_k})$ if the clusters $\mathcal{C}_{\mathbf{x}_i}$ are all distinct, and 0 otherwise (see Fig. 1). We expect that all these observables have fractal dimension $d_{T_{\min,k}}$ equal to the shortest-path fractal dimension $d_{\min}$. We aim to determine $\psi_{T_{\min,k}}$ for $k \geq 1$, as well as $d_{\min}$.

Finally, consider a pair of sites separated by a distance $\sim L$, say $\mathbf{x}=\mathbf{0}$ and $\mathbf{x}=\boldsymbol{\alpha}L$ where $\boldsymbol{\alpha} \in \mathbb{R}^d$, and let $S_{\boldsymbol{\alpha}L}$ be the

TABLE I. Numerical estimates versus theoretical predictions for exponents associated to $\langle C_{\min,2}\rangle$, $\langle M_{\min,2}\rangle$, and $S_{L/2}$.

Quantity	$q=$	0	0.01	0.25	0.5	1	1.5	2	3	3.5
$\langle C_{\min,2}\rangle$	p (num.)	1.25002(4)	1.1800(1)	0.9273(2)	0.8071(2)	0.6458(1)	0.5226(2)	0.4163(3)	0.214(2)	0.108(3)
	p (pred.)	1.25000	1.18007	0.92707	0.80768	0.64583	0.52298	0.41667	0.21667	0.10376
$\langle M_{\min,2}\rangle$	p (num.)	1.25002(4)	1.1639(5)	0.8527(5)	0.7036(3)	0.4994(7)	0.3444(4)	0.2087(2)	−0.0525(25)	−0.195(3)
	p (pred.)	1.25000	1.16439	0.85222	0.70341	0.50000	0.34420	0.20833	−0.05000	−0.19748
$S_{L/2}$	$d_{\min}$ (num.)	1.24999(3)	1.2371(10)	1.1825(3)	1.1596(4)	1.1303(8)	1.1112(7)	1.0955(10)	1.0677(40)	1.0560(30)
$2\le g\le 4$	$d_{\min}$ (conj.)	1.25000	1.23463	1.18211	1.15918	1.13021	1.10997	1.09375	1.06667	1.05343

length of the shortest path connecting these sites if one exists and 0 otherwise. Since $\mathbb{P}(\mathbf{0}\leftrightarrow \boldsymbol{\alpha}L)\sim L^{-2\beta/\nu}$, we expect that $\langle (S_{\boldsymbol{\alpha}L})^n\rangle \sim L^{nd_{\min}-2\beta/\nu}$. We will use $\langle (S_{\boldsymbol{\alpha}L})^n\rangle$ to numerically determine $d_{\min}$ in 2D, leading to conjecture (1).

IV. SCALING ARGUMENTS

Let $p_k(R)$ be the probability that, in an annulus of inner radius $r\sim O(1)$ and outer radius R, the inner circle is connected to the outer one by at least k distinct clusters. The k-arm exponent x_k characterizes the large-R asymptotics of $p_k(R)$ at criticality: $p_k(R)\sim R^{-x_k}$ as $R\to\infty$. In particular, $x_1=\beta/\nu$. In two dimensions it is known [9,11,20] that

$$x_1=(g-2)(6-g)/(8g), \tag{2a}$$

$$x_k=(g/8)k^2-(g-4)^2/(8g) \quad \text{for } k\ge 2. \tag{2b}$$

Now fix nearby sites $\mathbf{x}_1,\ldots,\mathbf{x}_k$, and let $C_k(s)$ be the probability that $\mathbf{x}_1,\ldots,\mathbf{x}_k$ belong to k distinct clusters, each of which contains at least s sites. The correspondence $s\sim R^{d_F}$ suggests that $C_k(s)\sim p_k(s^{1/d_F})$. Since $C_k(s)=\mathbb{P}(C_{\min,k}\ge s)$, we predict $\psi_{C_{\min,k}}=x_k/d_F+1$ and hence $\langle (C_{\min,k})^n\rangle\sim L^{nd_F-x_k}+\text{const}$. This agrees with standard hyperscaling for $k=1$; we will test it for $k\ge 2$. A similar argument suggests that $\langle B_s^n\rangle\sim L^{nd_F-x_2}$.

Analogous reasoning predicts $\psi_{M_{\min,k}}=x_k/d_H+1$ and $\psi_{T_{\min,k}}=x_k/d_{\min}+1$. For $q=1$ this argument for shortest-path scaling is due to Ziff [21]. In 2D it is known [9,20] that $d_H=1+2/g$, but to our knowledge Eq. (1) is the only known conjecture for $d_{\min}$, even for $q=1$.

V. MONTE CARLO SIMULATIONS

We simulated the two-dimensional random-cluster model at criticality for $q=0,0.01,0.25,0.5,1,1.5,2,3,3.5$ [22] and $4\le L\le 1024$ (periodic boundary conditions) by the Sweeny algorithm [14] when $0<q<1$ and the Chayes-Machta algorithm [23] when $q>1$. For $0.25\le q\le 2$ we also have data at $L=2048$. For $q=1$ we used cluster-growth algorithms to handle $4\le L\le 4096$. For $q=0$ we used Wilson's algorithm [24] to generate spanning trees from loop-erased random walk [25]. The total CPU time used in these simulations was approximately 66 years using a 3.2 GHz Xeon EM64T processor.

For each observable $\mathcal{O}$, we fit $\langle \mathcal{O}^n\rangle$ for $n=1,2,3$ to the ansatze aL^p, $aL^p+bL^{p-\Delta}$, and aL^p+c, varying $L_{\min}$ (the smallest L value included in the fit) until the χ^2 was reasonable. The error bar is a subjective 68% confidence limit that includes both statistical error and systematic error due to unincluded corrections to scaling.

Our results for $\langle C_{\min,2}\rangle$ and $\langle M_{\min,2}\rangle$ are presented in Table I; complete results for $\langle C^n_{\min,2}\rangle$, $\langle C^n_{\min,3}\rangle$ and $\langle M^n_{\min,2}\rangle$ with $n=1,2,3$ can be found in [26]. The agreement with the predicted exponents is excellent, except where the exponent is very negative and hence possibly overshadowed by correction-to-scaling terms. A finite-size-scaling plot for $C_{\min,2}$ and $q=1$ is shown in Fig. 2 and exhibits excellent collapse.

Next we studied $S_{\boldsymbol{\alpha}L}$ for $\boldsymbol{\alpha}=(\frac{1}{2},0)$ in order to estimate $d_{\min}$: see Table I and [26]. Our result for $q=1$ is compatible with Grassberger's [27] estimate $d_{\min}=1.1306(3)$. The corrections to scaling are very strong for these observables, and our error bars are dominated by our assessment of the likely systematic error from such corrections. It would be very useful (but also very expensive) to obtain data at larger values of L.

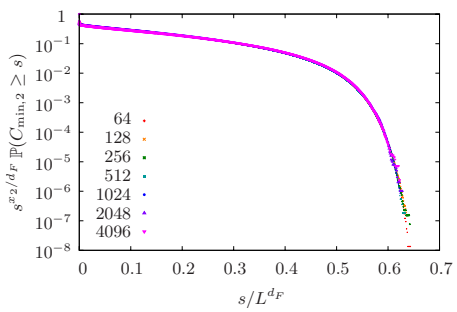

FIG. 2. (Color online) Finite-size-scaling plot showing $s^{x_2/d_F}\mathbb{P}(C_{\min,2}\ge s)$ versus s/L^{d_F} for $q=1$ and $64\le L\le 4096$.

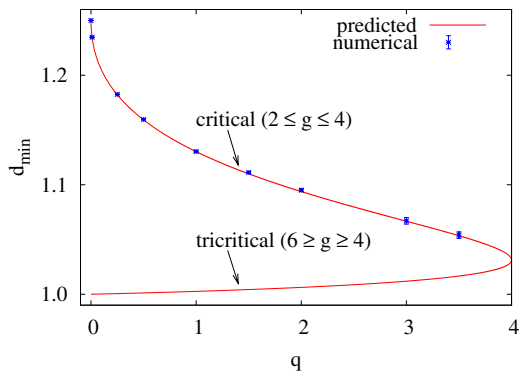

FIG. 3. (Color online) Numerical estimates for $d_{\min}$ together with the conjectured exact formula $d_{\min}\overset{?}{=}(g+2)(g+18)/(32g)$.

Our results for $d_{\min}$ are consistent with the simple formula (1): see Table I and Fig. 3. This formula has the nice property that $d_{\min}$ is monotone decreasing for $2 \le g \le 6$ and reaches $d_{\min}=1$ precisely at $g=6$, in accordance with the idea [28] that clusters become more compact as g grows. It also agrees with the known fact that $d_{\min}=\frac{5}{4}$ at $q=0$ [19]. Indeed, if one seeks a formula of the "Coulomb-gas" form $d_{\min}=F(g)=Ag+B+C/g$ [11] and imposes the constraints $F(2)=\frac{5}{4}$, $F(6)=1$, $F'(6)=0$, then the unique solution is Eq. (1).

Our results for $T_{\min,1}$ and $T_{\min,2}$ are similar to those shown in Table I and will be reported elsewhere.

ACKNOWLEDGMENTS

We are grateful to Gary Reich for a discussion that inspired this work; to Wenan Guo for participation in early stages of this project; to Russ Lyons, Robin Pemantle, and David Wilson for helpful correspondence; and to NYU ITS for use of their computer cluster. This work was supported in part by NSF Grant No. PHY-0424082, NSFC Grant No. 10975127, and the Chinese Academy of Sciences.

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[26] See supplementary material at http://link.aps.org/supplemental/10.1103/PhysRevE.81.020102 for full tables.

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